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LETTER TO THE EDITOR

Nonlinear stability analysis in multilayer quasigeostrophic systems

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Abstract. A sufficient condition for nonlinear stability of steady solutions of the quasigeostrophic equations in a multilayer system is found. It proves the stability of given shear flows in a two-layer model that were known to be neutrally stable only.

In this letter we study the problem of nonlinear stability of steady-state solutions in a multilayer quasigeostrophic fluid dynamical system. By a generalisation of a theorem (Arnol'd 1965) about nonlinear stability of stationary plane curvilinear flows, we obtain a sufficient condition for a steady solution to be stable in the Lyapunov sense, i.e. without linearising the evolution equations in the perturbation. The application of our result to a two-layer shear flow gives a rigorous stability condition which was previously obtained through linear computations.

The large scale motions of oceans and atmospheres are described (Pedlosky 1979) by the shallow-water equations in the quasigeostrophic approximation (small Rossby number). If the stratification in density is modelled by N superimposed layers whose constant densities are $\rho_1 < \rho_2 < \cdots < \rho_N$ (the first layer is the upper one) the evolution is given by the following system of partial differential equations (Pedlosky 1979)

$$\frac{\partial q_i}{\partial t} + J(\psi_i, q_i) = 0 \qquad i = 1, \ldots, N$$

$$J(a, b) = a_x b_y - a_y b_x \qquad + \text{boundary and initial conditions} \tag{1}$$

where ψ_i is the stream function of the ith layer (i.e. the horizontal velocity field in the ith layer is $\boldsymbol{v}_i(x, y, t) = (-\partial\psi_i/\partial y, \partial\psi_i/\partial x)$) and

$$q_i = \Delta\psi_i + F_i \sum_{j=1}^{N} T_{ij}\psi_j + f_i \qquad F_i = \frac{f_0^2}{g[(\rho_{i+1} - \rho_i)/\rho_0]D_i}$$

$$T_{ij} = \begin{pmatrix} -1 & 1 & & & & \\ 1 & -2 & 1 & & 0 & \\ & & \ddots & & & \\ & 0 & & 1 & -2 & 1 \\ & & & & 1 & -1 \end{pmatrix} \qquad f_i = f_0 + \beta y + \delta_{iN} b(y)$$

$$f_0 = 2\Omega \sin\varphi_0 \qquad \beta = 2\Omega\cos\varphi_0/R \qquad b = f_0 d(y)/D_N$$

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where g is the acceleration of gravity, ρ_0 the mean density, D_i the mean thickness of the ith layer, R the Earth's radius, Ω the Earth's angular velocity, φ_0 the reference latitude and $d(y)$ the bottom topography.

The steady solutions of equations (1) are given by

$$J(\bar{\psi}_i, \bar{q}_i) = 0 \qquad i = 1, \ldots, N \qquad + \text{boundary conditions.} \tag{2}$$

A solution of equations (2) is given by the equations

$$\bar{q}_i = P_i(\bar{\psi}_i) \qquad i = 1, \ldots, N \tag{3}$$

where P_i is any smooth function such that the boundary conditions can be satisfied. In a closed domain $\mathscr{C}$ the natural boundary conditions are

$$\psi_i|_{\partial\mathscr{C}} = c_i = \text{constant.}$$

However, in geophysical situations the open boundary conditions are imposed:

$$\lim_{(x,y)\to\pm\infty} \nabla\psi_i = 0. \tag{4}$$

Usually the study of stability of a steady solution of the differential problem

$$\frac{\partial f}{\partial t} = Af \qquad + \text{boundary and initial conditions} \tag{5}$$

(where A is a nonlinear operator) is carried out in the framework of the linearised stability theory (Joseph 1976). From a given steady solution $\bar{f}$ ($A\bar{f} = 0$) one can define the linear operator $\tilde{A}_{\bar{f}}$ by

$$A(\bar{f} + \delta f) = \tilde{A}_{\bar{f}}\,\delta f + \mathrm{O}(\delta f^2); \tag{6}$$

then the usual linearised stability theory consists in analysing the eigenvalues of the linear operator $\tilde{A}_{\bar{f}}$. If all the eigenvalues of $\tilde{A}_{\bar{f}}$ have a negative real part, the steady solution $\bar{f}$ of equation (5) is asymptotically stable. If one (or more than one) of them has a positive real part, $\bar{f}$ is unstable. If some eigenvalue is a pure imaginary number and the other eigenvalues are real negatives (neutral stability), no definitive conclusion can be obtained. This last case is, unfortunately, typical in inviscid fluid dynamical problems (see Joseph (1976) for a discussion of this point). Therefore, in this case, nonlinear techniques should be used to discuss stability.

In this context very few results are known and many of them are obtained by applying Lyapunov's well known second theorem. Let $\mathscr{T}$ be a metric space in which a time evolution T^t is given. $\bar{f} \in \mathscr{T}$ is a rest point if $T^t\bar{f} = \bar{f}$, $\forall t$. The above mentioned Lyapunov theorem states (Hahn 1963) the following.

Theorem. If a functional $H[f]$ exists such that (a) H has a relative minimum in $\bar{f}$ and (b)

$$\frac{\mathrm{d}}{\mathrm{d}t} H[T^t f] \leqslant 0 \qquad \text{if } f \in I_{\bar{f}}$$

where $I_{\bar{f}}$ is a neighbourhood of $\bar{f}$, then $\bar{f}$ is a stable rest point in the a Lyapunov sense. In this case H is called a Lyapunov functional. In our case a point f corresponds to a vector $\boldsymbol{\psi} \equiv (\psi_1, \ldots, \psi_N)$ and the time evolution T^t is determined by equation (1).

Let us emphasise that the explicit determination of a Lyapunov functional for a given system is very difficult and it is possible only in some cases in which there are particular symmetries or conservation laws. In our case, as an infinite number of conservation laws exists, it is possible to determine a Lyapunov functional for some steady solution of the problem.

Let us now present our main result: one can affirm that the steady solution $\bar{\boldsymbol{\psi}} = (\bar{\psi}_1, \ldots, \bar{\psi}_N)$ satisfying equations (3) and (4) with given functions P_i is stable in the Lyapunov sense if the following conditions hold:

$$P_i'(\bar{\psi}_i) > 0 \qquad i = 1, \ldots, N. \tag{7}$$

This sufficient condition for stability can be obtained as follows: defining

$$\Phi_i'(z) = P_i^{-1}(z) \qquad \text{i.e. } \Phi_i'(P_i(z)) = z$$

$$H[\boldsymbol{\psi}] = \int \left(\sum_{i=1}^{N} \frac{1}{F_i} [\tfrac{1}{2}(\nabla\psi_i)^2 + \Phi_i(q_i)] + \sum_{i=1}^{N-1} \tfrac{1}{2}(\psi_i - \psi_{i+1})^2 \right) \mathrm{d}x \, \mathrm{d}y$$

we want to prove that H is a Lyapunov functional for the steady solution $\bar{\boldsymbol{\psi}}$ under conditions (7); then, for the second Lyapunov theorem, $\bar{\boldsymbol{\psi}}$ is a stable steady solution. We thus have to prove that conditions (a) and (b) hold.

(a) The functional H has a minimum in $\bar{\boldsymbol{\psi}}$ because

$$\delta H[\bar{\boldsymbol{\psi}}] = 0 \qquad \delta^2 H[\bar{\boldsymbol{\psi}}] > 0$$

where δH and $\delta^2 H$ denote first and second variation of H. Indeed one has

$$\delta H[\boldsymbol{\psi}] = \int \left(\sum_{i=1}^{N} \frac{1}{F_i} (\nabla\psi_i \nabla\delta\psi_i + \Phi_i'(q_i)\delta q_i) + \sum_{i=1}^{N-1} (\psi_i - \psi_{i+1})(\delta\psi_i - \delta\psi_{i+1}) \right) \mathrm{d}x \, \mathrm{d}y$$

$$= \int \left(\sum_{i=1}^{N} \frac{1}{F_i} \Delta\delta\psi_i (\Phi_i'(q_i) - \psi_i) \right.$$

$$\left. + \sum_{i=1}^{N-1} (\delta\psi_i - \delta\psi_{i+1})(\psi_i - \Phi_i'(q_i) - \psi_{i+1} + \Phi_{i+1}'(q_{i+1})) \right) \mathrm{d}x \, \mathrm{d}y$$

$$\delta^2 H[\boldsymbol{\psi}] = \frac{1}{2} \int \left(\sum_{i=1}^{N} \frac{1}{F_i} [(\nabla\delta\psi_i)^2 + \Phi_i''(q_i)\delta q_i^2] + \sum_{i=1}^{N-1} (\delta\psi_i - \delta\psi_{i+1})^2 \right) \mathrm{d}x \, \mathrm{d}y.$$

The first variation is zero for $\boldsymbol{\psi} = \bar{\boldsymbol{\psi}}$ since $\bar{\psi}_i = \Phi_i'(\bar{q}_i)$; moreover $\delta^2 H$ is positive for $\boldsymbol{\psi} = \bar{\boldsymbol{\psi}}$ because of condition (7), which can also be written as

$$\Phi_i''(\bar{q}_i) > 0.$$

(b) The second condition of the theorem is verified because $H[\boldsymbol{\psi}]$ is conserved in time, i.e.

$$\frac{\mathrm{d}}{\mathrm{d}t} H[\boldsymbol{\psi}] = 0.$$

The functionals $\int \Phi_i(q_i) \, \mathrm{d}x \, \mathrm{d}y$ are time invariant for any $\Phi(\cdot)$ because of the conservation of the quantity q_i related to a fluid particle in the ith layer, expressed by (1). Besides, the total energy

$$E = \frac{1}{2} \int \left(\sum_{i=1}^{N} \frac{1}{F_i} (\nabla\psi_i)^2 + \sum_{i=1}^{N-1} (\psi_i - \psi_{i+1})^2 \right) \mathrm{d}x \, \mathrm{d}y$$

is also conserved. This can be seen by multiplying the ith equation in (1) by ψ_i/F_i and integrating:

$$0=\sum_{i=1}^{N}\frac{1}{F_i}\int\left(\psi_i\frac{\partial q_i}{\partial t}+\psi_i J(\psi_i, q_i)\right)\mathrm{d}x\,\mathrm{d}y$$

$$=\sum_{i,j=1}^{N}\int\frac{1}{F_i}\psi_i\frac{\partial}{\partial t}(\Delta\psi_i+F_iT_{ij}\psi_j)\,\mathrm{d}x\,\mathrm{d}y$$

$$=-\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\left(\sum_{i=1}^{N}\frac{1}{F_i}(\nabla\psi_i)^2+\sum_{i=1}^{N-1}(\psi_i-\psi_{i+1})^2\right)\mathrm{d}x\,\mathrm{d}y.$$

The invariance of H is thus proved, so $\bar{\psi}$ is stable if conditions (7) are verified.

As an application let us consider a shear flow described in a two-layer system by the steady solution

$$\bar{\psi}_i=U_iy \qquad i=1,2 \qquad U_i>0. \tag{8}$$

This solution does not satisfy the boundary condition (4). However, the unmodified conditions for stability (7) hold also in this case providing the domain is a cylinder with its axis parallel to the earth's rotation axis; the damping of the velocity at $y\to\pm\infty$ must also be prescribed.

$$\begin{aligned}P_1'(\bar{\psi}_1)&=\frac{\nabla(\Delta\bar{\psi}_1+F_1(\bar{\psi}_2-\bar{\psi}_1)+\beta y)}{\nabla\bar{\psi}_1}>0\\ P_2'(\bar{\psi}_2)&=\frac{\nabla(\Delta\bar{\psi}_2+F_2(\bar{\psi}_1-\bar{\psi}_2)+\beta y+b)}{\nabla\bar{\psi}_2}>0.\end{aligned}\tag{9}$$

Then if the velocities satisfy

$$-\frac{\beta}{F_1}<U_2-U_1<\frac{B}{F_2} \qquad B=\beta+b' \tag{10}$$

stability is ensured rigorously, i.e. without having linearised the equations of motion.

If U_1, $U_2<0$, i.e. if the velocities are directed eastward, conditions (9) can be applied in a moving reference frame. Let us make a Galilean transformation

$$x=x'+ct \qquad y=y' \qquad t=t'$$

with $c>\max(|U_1|,|U_2|)$ and $c\ll R/T$ where T is the Earth's period of rotation; equations (1) become

$$\frac{\partial\tilde{q}_i}{\partial t'}+J'(\tilde{\psi}_i,\tilde{q}_i)=0$$

where

$$\tilde{\psi}_i=\psi_i'+cy' \qquad \tilde{q}_i=\Delta\tilde{\psi}_i+F_i\sum_{j=1}^{N}T_{ij}\tilde{\psi}_j+\beta'y'+\delta_{i2}b$$

with $\beta'\simeq\beta$ because $c\ll R/T$.

In this new frame our steady solution is

$$\tilde{\psi}_i=(c-U_i)y'=U_i'y' \qquad U_i'>0$$

and so our stability condition can be applied. One obtains

$$-\beta/F_1 < U_1 - U_2 < \beta/F_2. \tag{11}$$

Conditions (10) and (11) have an immediate physical meaning.

In order to study the stability properties of the steady solution (8), Pedlosky (1964), Gill *et al* (1974) and Tang (1979) linearised the evolution equations in the perturbation $\delta\psi$ and obtained that,under the same condition (11), $\delta\psi(x, y, t)$ remains bounded but does not decrease in time; thus stability was not inferred rigorously, as discussed at the beginning of this letter. Here, on the contrary, condition (11) (and (10) for $U_i > 0$) is a sufficient condition for rigorous stability (in the Lyapunov sense) of $\bar{\psi}_i$.

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